Ma2a Practical – Recitation 4

Fall 2024

Exercise 1. (Bernoulli equations)
Consdier the equation

$$y' + p(t)y = q(t)y^n$$

1. Solve the equation when $n = 0, 1$.
2. Show that if $n \neq 0, 1$, the substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation.

Exercise 2. Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = e^{\int Q(y)dy}.$$

For example, consider the case when $M = xy + y^2$ and $N = x^2 + 3xy$, then $Q = \frac{1}{y}$.

Exercise 3. Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x}$$

1. Show that the equation is homogeneous. [1]
2. Introduce a new variable v such that $v = \frac{y}{x}$. Rewrite the above equation in terms of v and x and solve it: (We can keep the integral)

$$v + x\frac{dv}{dx} = v - 4$$

[1]The differential equation $\frac{dy}{dx} = f(x, y)$ is said to be homogeneous if $f(x, y)$ can be expressed as a function of the ration $\frac{y}{x}$.

Solution

Notice that if $n = 0$ or $n = 1$ then the equation is linear and we already know how to solve the first order linear differential equation.

When $n \geqslant 2$, we will first divide the differential equation by y^n to get,

$$y^{-n}y^{'} + p(t)y^{1-n} = q(t) \tag{1}$$

We are now going to use the substitution $v = y^{1-n}$ to convert this into a differential equation in terms of v. By chain rule, we have:

$$v^{'} = (1-n)y^{-n}y^{'}$$

Now, plugging this into eq.1. we get:

$$\frac{1}{1-n}v^{'} + p(t)v = q(t) \tag{2}$$

We can solve v since eq.2 is a first order linear differential equation. Then we can also get the solution to the original differential equation by plugging v back into our substitution and solving for y.

Solution

A function $\mu(x, y)$ is an integrating factor if the equation $\mu M + \mu N y' = 0$ is exact, *i.e.* if $\partial_y(\mu M) = \partial_x(\mu N)$. Expanding this equation, we obtain

$$\partial_y \mu M + \mu(\partial_y M - \partial_x N) = \partial_x \mu N.$$

This equation is too hard to solve in general. The problem provides the extra assumption $\partial_x N - \partial_y M = Q(y)M$, and we will find a solution using this assumption. Then the equation becomes

$$M(\partial_y \mu - Q(y)\mu) = \partial_x \mu N.$$

We further simplify the equation by assuming that μ depends only on y, so $\partial_x \mu = 0$. The equation is then

$$M(\mu' - Q\mu) = 0,$$

and we see that $\mu(y) = \exp\left(\int Q\right)$ is a solution. Thus this μ is an integrating factor.

In the case $M = xy + y^2$ and $N = x^2 + 3xy$, $Q = \frac{N_x - M_y}{M} = \frac{x+y}{xy+y^2} = \frac{1}{y}$. Then we choose $\mu = e^{\int_1^y Q} = y$ for $y > 0$. We then look for a function $\Psi(x, y)$ such that

$$\partial_x \Psi = y(xy + y^2) = xy^2 + y^3$$
$$\partial_y \Psi = y(x^2 + 3xy) = x^2 y + 3xy^2.$$

The first equation gives $\Psi(x, y) = \frac{1}{2}x^2y^2 + xy^3 + K(y)$, where K is an unknown function. Plugging back into the second equation gives $K'(y) = 0$ so K is a constant, and we may choose it to be 0. The solutions are then described by the implicit equation:

$$\frac{1}{2}x^2y^2 + xy^3 = C.$$

Solution

Recall that the differential equation $\frac{dy}{dx} = f(x, y)$ is said to be homogeneous if $f(x, y)$ cna be expressed as a function of the ration $\frac{y}{x}$. Now our equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y} = \frac{(y - 4x)/x}{x/x} = \frac{y}{x} - 4$$

Therefore the equation is homogeneous.

Let

$$v = \frac{y}{x}$$

We have $v' = \frac{y'}{x} - \frac{y}{x^2}$, or equivalently $xv' + v = y'$. The equation becomes:

$$v + xv' = v - 4 \Leftrightarrow v' = -\frac{4}{x}.$$

Separate the variable:

$$dv = -4\frac{dx}{x}$$

Integrate both side, we have:

$$v = -4 \ln|x| + c$$

So

$$y = xv = -4x \ln|x| + cx.$$